Grasping rules and semiclassical limit of the geometry in the Ponzano–Regge model

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We show how the expectation values of geometrical quantities in 3d quantum gravity can be explicitly computed using grasping rules. We compute the volume of a labelled tetrahedron using the triple grasping. We show that the large spin expansion of this value is dominated by the classical expression, and we study the next to leading order quantum corrections.

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I. INTRODUCTION

The spinfoam non–perturbative approach to quantum gravity [1] describes a microscopical quantum geometry, where the geometrical observables have discrete values, expressed in terms of half–integers (spins). These spins characterize "atoms" of spacetime. A possible way to study the semiclassical limit consists in studying the expansion for large values of the spins. In this paper, we apply this procedure to the volume of a tetrahedron in the Ponzano–Regge (PR) model for 3d Riemannian quantum gravity [2].

The spins, which are the fundamental variables of the model, label the irreducible representations (irreps) of SU(2), the gauge group of 3d GR. Using the generating functional introduced in [3], the geometrical expectation values can be defined via the action of grasping operators. This action can be evaluated using the recoupling theory of SU(2), and thus the geometrical values given in terms of purely algebraic quantities. Here we show that the relevant graspings can be identified starting from the discretization of the classical observables, and we focus on the first non–trivial observable, the quantum volume of a tetrahedron. Its value is obtained by triple graspings acting on the $\{6j\}$ symbol associated with the tetrahedron. We identify all the relevant graspings, and evaluate their action to write explicitly the value of the quantum volume in terms of algebraic quantities. This is our first result. The relevance of this result concerns the construction of spinfoam models of matter coupled to quantum gravity. The matter action typically contains a volume term, and thus knowing the value of the quantum volume of a labelled tetrahedron is needed for constructing the coupled model. For instance, this result can be used in [5] and [6]. Furthermore, a power series of triple graspings acting on the SU(2) $\{6j\}$ symbol can be used to reconstruct the quantum $\{6j\}$ symbol used in the Turaev–Viro model [4].

Studying the large spin expansion of this value, we identify the dominant and subdominant graspings, and show that the expansion is remarkably dominated by the classical formula. However, the exact result for the volume has an extra factor multiplying the classical formula. This factor can be understood in terms of the well–known feature of the PR model to sum over both orientations of spacetime. The consequences of this fact for the volume were discussed in [4], and the factor here obtained agrees with their results. Motivated by the analysis of [4], we then consider the *squared* volume, and show that the correct classical formula is this time directly reproduced. This is our second result. The relevance of this result is to support the idea that the semiclassical limit of spinfoam models can be studied considering the large spin expansion. This idea is also supported by the recent calculations of the 2-point function [7, 8, 9]. This is the main open problem in the formalism, and other remarkable ideas to address it include defining coarse graining procedures [10], contructing effective field theories [11], or rewriting conventional quantum field theories in the language of spinfoams [12].

This paper is organized as follows. In Section II, we briefly recall how the Ponzano–Regge model is constructed, and we introduce the generating functional to compute expectation values of the geometry. In Section III, we show how to introduce the geometrical observables using the discrete variables of the model, and how to relate this observables to grasping operators. We compute the values of quantum lengths and angles, corresponding to quadratic graspings, and the value of the quantum volume, corresponding to the triple grasping. In Section IV, we analyze the leading order of the large spin expansion for the volume and the squared volume. In Section V, we compute the next to leading order quantum corrections. In the final Section VI we summarize our results. All the calculations used in the paper

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are explicitly reported in the Appendix, where we also fix our phase convention for the evaluation of spin networks, and show how this is related to the grasping rules.

Throughout we define the Planck length as $\ell_{\rm P} := 16\pi\hbar G$.

II. PONZANO–REGGE MODEL AND GENERATING FUNCTIONAL

We consider here the PR model on a fixed triangulation Δ of the spacetime manifold M. This can be obtained from a path integral quantization of the first order triad action for Riemannian GR. In the continuum, the fundamental fields are the SU(2) connection ω_μ^{IJ} and the triad e_μ^I, and the action for GR reads

$$S_{\rm GR}[e_\mu^I(x), \omega_\mu^{IJ}(x)] = \frac{1}{16\pi G}\int_M \mathrm{Tr}\; e\wedge F(\omega), \tag{1}$$

where the trace Tr is over the SU(2) indices, and $F(\omega)$ is the curvature of ω. This action can be discretised on Δ as follows. Consider an embedding $i:\Delta\to M$, which allows to think of Δ as a cellular decomposition of M. Using this embedding, we can define the vectors $\ell_s^\mu \sim \int_s dx^\mu$, tangent to the segments s of Δ. Analogously, we can define the vectors $\ell_e^\mu \sim \int_e dx^\mu$ tangent to the edges e of the dual triangulation Δ^* (which we recall are in one to one correspondence with triangles of Δ).

We then discretise the triad field with an $\mathfrak{su}(2)$ algebra element, associated to the segments of Δ, and the connection with an SU(2) element, associated to the edges of Δ^*:

$$e_\mu^I(x) \mapsto X_s^I \;:=\; \frac{1}{\ell_{\rm P}} e_\mu^I(x)\ell_s^\mu \sim \frac{1}{\ell_{\rm P}}\int_s e_\mu^I(x)dx^\mu. \tag{2}$$

$$\omega_\mu^{IJ}(x) \mapsto g_e \;:=\; e^{\omega_\mu^{IJ}\ell_e^\mu} \sim e^{\int_e \omega^{IJ}}. \tag{3}$$

We also introduce the quantities $g_f = \prod_{e\in\partial f} g_e$ associated with the faces f of Δ^*. We see from (3) that these quantities discretise the curvature, $g_f \sim \exp\int_f F(\omega)$. Recalling that faces of Δ^* are in one to one correspondence with segments of Δ, we will use from now on the notation $g_s \equiv g_f$.

Consequently, we write the discrete action as

$$S[X_s^I, g_e] = \hbar\sum_s \mathrm{Tr}\,[X_s\, g_s]. \tag{4}$$

When the embedding is sufficiently refined, and the coordinate areas consequently small, we can expand $g_s \sim \mathbb{1} + \int_f F(\omega)$. Recalling that Tr $T = 0$ for any $T\in\mathfrak{su}(2)$, we see that (4) reduces to (1).

The quantum theory can be constructed from the partition function

$$Z = \prod_e \int_{\rm SU(2)} dg_e \prod_s \int_{\mathfrak{s}u(2)} dX_s\; e^{i\sum_s \mathrm{Tr}\,[X_s\, g_s]}. \tag{5}$$

This quantity can be evaluated using the harmonic analysis of SU(2) (for details, see for instance [1]), and one obtains

$$Z = \sum_{j_s}\prod_s d_{j_s}\prod_\tau \{6j\}, \tag{6}$$

where the sum is over all possible assignments of half–integers j to the segments of Δ. The half–integers, or spins, label the irreducible representations of SU(2). The quantity $d_j := 2j+1$ is the dimension of the representation. Finally, a $\{6j\}$ symbol is associated with each tetrahedron τ of Δ. For more discussion of the PR model, see [13]. The $\{6j\}$ symbol is the key object of the recoupling theory of SU(2), and it depends only on the six js attached to the segments of the tetrahedron. It can be written in terms of the Wigner 3m–symbols as

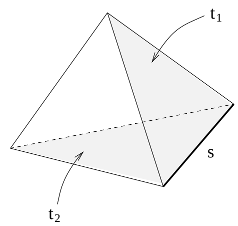
$$\left\{\begin{array}{ccc} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{array}\right\} := \left(\begin{array}{ccc} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{array}\right)\left(\begin{array}{ccc} j_1 & j_5 & j_6\\ m_1 & m_5 & m_6\end{array}\right)\left(\begin{array}{ccc} j_4 & j_5 & j_3\\ m_4 & m_5 & m_3\end{array}\right)\left(\begin{array}{ccc} j_4 & j_2 & j_6\\ m_4 & m_2 & m_6\end{array}\right). \tag{7}$$

The $\{6j\}$ symbol has a well–known asymptotic behaviour, namely that if we rescale the half–integers entering the $\{6j\}$ symbol as $j_s \equiv N k_s$, we have [2, 14, 15]

$$\lim_{N\mapsto\infty}\{6j\} \;=\; \frac{1}{\sqrt{12\pi V(j_s)}}\cos\left(\sum_s (j_s+\frac{1}{2})\theta_s(j_s)+\frac{\pi}{4}\right), \tag{8}$$

where $V(j_s)$ is the classical volume of the tetrahedron with segment lengths given by $j+\frac{1}{2}$, and θ_s are the corresponding dihedral angles, namely the angles between the normals to the triangles. As it will be useful in the following, we recall here that the volume of a tetrahedron can be expressed in terms of a dihedral angle using the formula

$$V = \frac{2}{3}\frac{A_1A_2}{\ell_s}\sin\theta_s, \tag{9}$$



where A_1 and A_2 are the areas of the two triangles t_1 and t_2 sharing the segment s, as shown in the above figure. The areas can be expressed in terms of the segment lengths, using Heron's formula (see the Appendix).

The key point of (8) is the argument of the cosine: up to the factor $\frac{\pi}{4}$ (which does not change the equations of motion), this is the Regge action $S_{\rm R} = \sum_s \ell_s\theta_s(\ell_s)$, a discrete approximation to classical GR. Therefore, the amplitude of Z is dominated by exponentials of the Regge action, which makes it promising to study the semiclassical limit in this way. If this is correct, then also the geometrical quantities that one can evaluate in the model should reduce to their classical expressions, in the large j limit defined above.

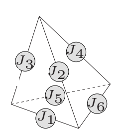
However one would expect a single exponential to arise in the semiclassical limit, namely $Z_{\rm GR} \sim \int \mathcal{D}g\, e^{iS_{\rm R}}$. The meaning of this difference is well studied, see for instance the discussion in [1]. We might then conclude that on a single tetrahedron τ $Z(\tau) = Z_{\rm GR}(\tau) + \overline{Z_{\rm GR}(\tau)}$. As we will see below, this fact has consequences for the expectation value of the volume.

To introduce the geometrical observables in the PR model, below we write them as gauge–invariant functions of the variables X_s^I. Then, to compute their expectation values, it is convenient to introduce a generating functional,

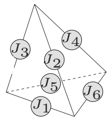
$$Z[J] = \prod_e \int_{\mathrm{SU}(2)} dg_e \, \prod_s \int_{\mathfrak{su}(2)} dX_s \; e^{i\sum_s \mathrm{Tr}\, X_s(g_s+J_s)}. \tag{10}$$

This can be evaluated as described in [3], to give

$$Z[J] = \sum_{\{j_s\}} \prod_s d_{j_s} \; \prod_\tau \quad , \tag{11}$$



where



$$:= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_5 & j_6 \\ n_1 & m_5 & m_6 \end{pmatrix}\begin{pmatrix} j_4 & j_5 & j_3 \\ m_4 & n_5 & n_3 \end{pmatrix}\begin{pmatrix} j_4 & j_2 & j_6 \\ n_4 & n_2 & n_6 \end{pmatrix}\times$$

$$D^{(j_1)}_{m_1n_1}(e^{J_1})D^{(j_2)}_{m_2n_2}(e^{J_2})D^{(j_3)}_{m_3n_3}(e^{J_3})D^{(j_4)}_{m_4n_4}(e^{J_4})D^{(j_5)}_{m_5n_5}(e^{J_5})D^{(j_6)}_{m_6n_6}(e^{J_6}). \tag{12}$$

Here the D are representation matrices. The quantity (12) represents the $\{6j\}$ symbols with source insertions. The J's are attached to the segments of Δ; they are the sources of the quantum excitations j_s. For all $J_s = 0$ (12) reduces to the expression for the $\{6j\}$ symbol given above, thus $Z_{\rm BF} = Z[J]\Big|_{J=0}$.

Consider now a gauge–invariant observable constructed from the X_s^I variables, $\Phi[X_s^I]$. Using the generating functional, its expectation value can be written as

$$\langle\Phi\rangle = \prod_e \int_{\mathrm{SU}(2)} dg_e \prod_s \int_{\mathfrak{su}(2)} dX_s\ \Phi[X_s^I]\ e^{i\sum_s X_s^I g_s^I} = \Phi[-i\frac{\delta}{\delta J_s^I}]Z[J]\Big|_{J=0}. \tag{13}$$

In the next Section, we will use this procedure to compute expectation values. To do so, we need to know the action of the algebra derivatives. Acting on a group element in the representation j, we have

$$\frac{\delta}{\delta J^I} D^{(j)}(e^J)\Big|_{J=0} = -iT^{I(j)}, \tag{14}$$

where $T^{I(j)}$ is the I-th generator in the representation j. By inspecting (12), we see that a derivative acting on J_s attaches to the segment s an algebra generator in the irrep j_s labeling the segment. This action is called "grasping", and it is described in more details in the Appendix. In particular, we are interested in quadratic and cubic gauge–invariant functions Φ, such as squared lengths and volumes, which are related to the action of quadratic and cubic graspings.

III. GRASPINGS AND QUANTUM GEOMETRY

A. Quadratic graspings: lengths and angles

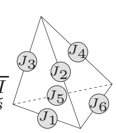
The classical geometrical observables can be described using the discrete variables X_s^I. For instance, the length of a segment $\tilde{s}$ can be written as $\ell_{\tilde{s}}^2 = g_{\mu\nu}\ell_{\tilde{s}}^\mu \ell_{\tilde{s}}^\nu = \ell_{\mathrm{P}}^2 X_{\tilde{s}}^I X_{\tilde{s}}^I$. In the same way, we can study the angles between the segments. To this aim, we consider the two segments s_1 and s_2, sharing a vertex, and the segment s_3 closing the triangle. The angle can be read from the scalar product $\ell_{s_1}\cdot\ell_{s_2} = \ell_{\mathrm{P}}^2 X_{s_1}^I X_{s_2}^I$.

Calculating the expectation value of quadratic functions of the X_s^Is is particularly simple. The expectation value of a scalar product is given by

$$\langle \ell_{s_1}\cdot\ell_{s_2}\rangle = \frac{1}{Z}\ell_{\mathrm{P}}^2 \prod_e \int_{\mathrm{SU}(2)} dg_e \prod_s \int_{\mathfrak{su}(2)} dX_s\ X_{s_1}^I X_{s_2}^I\ e^{i\sum_s X_s^I g_s^I} = -\frac{1}{Z}\ell_{\mathrm{P}}^2\ \frac{\delta}{\delta J_{s_1}^I}\frac{\delta}{\delta J_{s_2}^I}\ Z[J]\ \Big|_{J=0}. \tag{15}$$

To evaluate this quantity, we need the action of the double grasping, which is computed in the Appendix.

For the case $s_1 = s_2 \equiv \tilde{s}$, the relevant grasping gives

$$\frac{\delta}{\delta J_{\tilde{s}}^I}\frac{\delta}{\delta J_{\tilde{s}}^I}$$


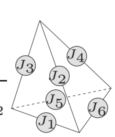
$$= -C^2(j_{\tilde{s}})\,\{6j\}. \tag{16}$$

Consequently, we have

$$\langle \ell_{\tilde{s}}^2\rangle = \frac{1}{Z}\sum_{\{j_s\}} \ell_{\mathrm{P}}^2 C^2(j_{\tilde{s}})\ \prod_s d_{j_s}\ \prod_\tau \{6j\} \tag{17}$$

We see that the value of the quantum squared length of a labelled segment is given by $\ell_{\mathrm{P}}^2 C^2(j_{\tilde{s}})$. This is the basic result of quantum geometry, and we wee that the origin of its discreteness lies in the compactness of the group SU(2), which gives a discrete series of representations. Therefore, ℓ_s acquires only discrete values, consistently with the canonical result [1].

For the case when $s_1 \neq s_2$ share a vertex, the relevant grasping gives

$$\frac{\delta}{\delta J_{s_1}^I}\frac{\delta}{\delta J_{s_2}^I}$$


$$= -\frac{1}{2}[C^2(j_{s_1}) + C^2(j_{s_2}) - C^2(j_{s_3})]\,\{6j\}_\tau, \tag{18}$$

where s_3 closes the triangle with s_1 and s_2. Therefore, we see that the value of $\cos\theta_{s_1 s_2}$ is given by the expression

$$\frac{1}{2C(j_{s_1})C(j_{s_2})}[C^2(j_{s_1}) + C^2(j_{s_2}) - C^2(j_{s_3})]. \tag{19}$$

Notice that this expression coincides with the classical one, once we identify the Casimir with the segment lengths. However, this does not mean that the angle behaves classically: as the Casimir is discrete, the angle cannot range continously between 0 and π, instead only specific values are allowed. Among these, the equilateral case: if we set all $j_s = j_0$ in (19), we obtain $\frac{1}{2} = \cos(\frac{\pi}{3})$, which correctly reproduces an equilateral triangle.

B. Triple grasping: volume

We now come to the explicit computation of the volume, which is the first non trivial geometrical object in the PR model. This is defined as

$$\mathcal{V}_\tau := \int_\tau e d^3x \simeq \frac{1}{3!} e \int_\tau d^3x, \qquad e = \frac{1}{3!}\epsilon^{\mu\nu\rho}\epsilon_{IJK} e^I_\mu e^J_\nu e^K_\rho. \tag{20}$$

The reason for the factor 3! lies in the fact the the determinant is the infinitesimal (metrical) volume of a cube, and there are 6 tetrahedra in a cube. To express $\mathcal{V}_\tau$ in terms of the variables X^I_s, we proceed as follows. Consider three segments sharing a point p of τ, with coordinate vectors $\ell^\mu_s(p)$; the coordinate volume is $\int_\tau d^3x = \det \ell^\mu_s(p)$, by definition of $\ell^\mu_s(p)$, independently of the point considered. We can think of $\ell^\mu_s(p)$ as a 3 by 3 matrix, with inverse $n^s_\mu(p)$ being defined by

$$\sum_{s\in p(\tau)} \ell^\mu_s(p) n^s_\nu(p) = \delta^\mu_\nu. \tag{21}$$

This resolution of the identity can now be inserted in the definition (20) of e, in order to give the variables X^I_s:

$$\mathcal{V}_\tau = \frac{1}{3!} e \det \ell^\mu_s = \frac{\det \ell^\mu_s}{3!}\frac{\ell_{\mathrm{P}}^3}{3!}\sum_{s_1\in p_1}\sum_{s_2\in p_2}\sum_{s_3\in p_3} \epsilon_{IJK}\,\epsilon^{\mu\nu\rho}\, n^{s_1}_\mu n^{s_2}_\nu n^{s_3}_\rho\, X^I_{s_1} X^J_{s_2} X^K_{s_3}.$$

Notice that from the definition of $n^s_\mu(p)$ in (21), we have $\epsilon^{\mu\nu\rho} n^{s_1}_\mu n^{s_2}_\nu n^{s_3}_\rho = \epsilon^{s_1 s_2 s_3}(\det \ell^\mu_s)^{-1}$, thus

$$\mathcal{V}_\tau = \frac{\ell_{\mathrm{P}}^3}{3!^2}\sum_{s_1\in p_1}\sum_{s_2\in p_2}\sum_{s_3\in p_3} \epsilon_{IJK}\,\epsilon^{s_1 s_2 s_3}\, X^I_{s_1} X^J_{s_2} X^K_{s_3}.$$

With an eye at the construction of its quantum version, it is convenient to symmetrise this expression. One possible way to do so is to take the same point p in each insertion of (21), and then sum over the four contributions:

$$\mathcal{V}_\tau = \frac{\ell_{\mathrm{P}}^3}{3!^2}\frac{1}{4}\sum_{p\in\tau}\sum_{s_1,s_2,s_3\in p} \epsilon_{IJK}\,\epsilon^{s_1 s_2 s_3}\, X^I_{s_1} X^J_{s_2} X^K_{s_3} = \frac{\ell_{\mathrm{P}}^3}{3!}\frac{1}{4}\sum_{p\in\tau}\epsilon_{IJK}\, X^I_{s_1} X^J_{s_2} X^K_{s_3}, \tag{22}$$

where in the latter expression s_1, s_2 and s_3 are a fixed right–handed triple belonging to the given p.

Alternatively, we can consider all possible sixteen non–coplanar triples of segments, and write

$$\mathcal{V}_\tau = \frac{1}{3!}\frac{\ell_{\mathrm{P}}^3}{16}\sum_{s_1,s_2,s_3} \epsilon_{IJK}\, X^I_{s_1} X^J_{s_2} X^K_{s_3}. \tag{23}$$

This formula was proposed in [3]. Notice that this latter case is more generic: (22) can be obtained from (23) restricting the triplets in the sum. The two expressions are clearly classically equivalent, but lead to different quantum values. As we show below, the corresponding values in the quantum theory have the same semiclassical leading order, but different corrections.

As it is more generic, we consider first the expression (23). Using the generating functional as above, we have

$$\begin{aligned}\langle \mathcal{V}_{\tilde\tau}\rangle &= \frac{1}{Z}\,\frac{1}{3!}\,\frac{\ell_{\rm P}^3}{16}\prod_e \int_{\rm SU(2)} dg_e \prod_s \int_{\mathfrak{su}(2)} dX_s \sum_{s_1,s_2,s_3} \epsilon_{IJK} X^I_{s_1} X^J_{s_2} X^K_{s_3}\, e^{i\sum_s X_s^I g_s^I} = \\ &= \frac{1}{Z}\,\frac{i}{3!}\,\frac{\ell_{\rm P}^3}{16}\left(\sum_{s_1,s_2,s_3}\epsilon_{IJK}\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^J_{s_2}}\frac{\delta}{\delta J^K_{s_3}}\right) Z[J]\,\Big|_{J=0}. \end{aligned} \tag{24}$$

The derivatives act only on the sources present in the tetrahedron $\tilde\tau$. Therefore, $\langle \mathcal{V}_{\tilde\tau}\rangle$ depends only on the six js entering $\tilde\tau$. The quantity in round brackets gives rise to the triple grasping on the tetrahedron, as explained in the Appendix. Depending on the configuration of the triplet s_1, s_2, s_3 considered, different types of grasping are involved. These types, together with the result of their evaluation, are listed in Table I. We report the details of the evaluation in the Appendix, and discuss here the results. First of all, notice the colour coding: we used red for those graspings, type 2 and 3, corresponding to non coplanar triplets of different segments, which would be classically used to compute the volume. Blue, on the other hand, is used for configurations which classically would not contribute. These correspond to cases when the three segments are coplanar (type 1), or when two (type 4 and 5) and even all three (type 6) are the same. As in the classical case, coplanar triplets do not contribute to the volume (and this is the reason why we did not include this grasping in (23)). On the contrary, the classically absent types 4, 5 and 6 do contribute to the volume, but only at the level of quantum corrections, as they scale as j^2 (see column on the right of the Table). Indeed, the relevant graspings for the semiclassical limit are only the types 2 and 3, as they scale as j^3.

We see from the results listed in the table that evaluating the action of the triple grasping is more involved than the quadratic one. In particular, notice that the graspings 2, 3 and 5 do not preserve the original tetrahedral amplitude $\{6j\}$, but contain a superposition of terms.

The results listed in the Table can be extended to the other configurations by permutating the segment labels. For instance, we reported the grasping of type 2 acting on the vertex 126. The evaluation of the same grasping on, say, the vertex 234 can be obtained under the permutation $123456 \mapsto 231564$. With this understanding, we can write the action of (22) as

$$\frac{i}{3!}\frac{\ell_{\rm P}^3}{4}\sum_{p\in\tau}\left[c_-(j_s)\begin{Bmatrix} j_1-1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix} + c_0(j_s)\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix} + c_+(j_s)\begin{Bmatrix} j_1+1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}\right]\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}^{-1}, \tag{25}$$

where the explicit values of the coefficients $c_\pm(j_s)$, $c_0(j_s)$ are given in the Appendix. Notice that they vary when different configurations (here different points of the tetrahedron) are considered. For simplicity of notation we did not write explicitly the dependences of the coefficients on the configuration.

Type 2 is the only grasping entering (22), whereas the more generic definition (23) involves all the different graspings, and the result would be (i times) the sum of all the grasping evaluations listed in the Table. Notice that in both cases the value of the volume is purely imaginary. This was anticipated in [4], and can be understood as follows. Firstly, the volume is odd under change of orientation, namely $V(\tau) = -V(-\tau)$. Secondly, the unitarity of the PR amplitude implies $Z(\tau) = \overline{Z(-\tau)}$. Therefore $\langle V(\tau)\rangle = -\overline{\langle V(\tau)\rangle}$ which implies $\mathrm{Re}\langle V(\tau)\rangle \equiv 0$. As argued in [4], this volume can be related to a real volume $\langle V(\tau)\rangle_{\rm GR}$, computed using the $Z_{\rm GR}$ partition function defined in Section II, which we recall satisfies $Z = 2\,\mathrm{Re}\, Z_{\rm GR}$. Formally we have

$$\langle V(\tau)\rangle Z(\tau) = \langle V(\tau)\rangle_{\rm GR} Z_{\rm GR}(\tau) + \overline{\langle V(\tau)\rangle_{\rm GR} Z_{\rm GR}(\tau)} = 2i\langle V(\tau)\rangle_{\rm GR}\,\mathrm{Im}\, Z_{\rm GR}(\tau), \tag{26}$$

from which

$$\langle V(\tau)\rangle = i\,\frac{\mathrm{Im}\, Z_{\rm GR}(\tau)}{\mathrm{Re}\, Z_{\rm GR}(\tau)}\,\langle V(\tau)\rangle_{\rm GR} \tag{27}$$

Let us now restrict this formula to a single tetrahedron, and assume that in the semiclassical limit $Z_{\rm GR}(\tau) \sim \exp i(S_{\rm R}(\tau) + \frac{\pi}{4})$. Under this assumption, the large spin limit of (27) formally reads

$$\langle V(\tau)\rangle = i\tan\left(S_{\rm R}(j_s) + \frac{\pi}{4}\right)\langle V(\tau)\rangle_{\rm GR}. \tag{28}$$

We expect the value of the volume in the PR model to obey a relation like the one above. Also, notice that the same reasoning applied to the *squared* volume gives $\langle V(\tau)^2\rangle = \langle V(\tau)^2\rangle_{\rm GR}$, thus the PR model should indeed reproduce the correct semiclassical limit of the squared volume.

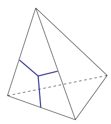
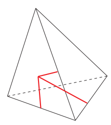
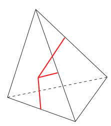
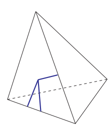
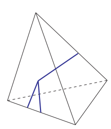
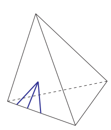
Grasping	Evaluation	Leading order
1. 	0	0
2. 	$c_-(j_s)\left\{\begin{array}{ccc} j_1-1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}+c_0(j_s)\left\{\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}+c_+(j_s)\left\{\begin{array}{ccc} j_1+1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}$	j^3
3. 	$c_-(j_s)\left\{\begin{array}{ccc} j_1-1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}+c_0(j_s)\left\{\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}+c_+(j_s)\left\{\begin{array}{ccc} j_1+1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}$	j^3
4. 	$-\frac{1}{2}\left[C^2(j_1)+C^2(j_2)-C^2(j_3)\right]\left\{\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}$	j^2
5. 	$-\frac{c_-(j_s)}{j_1+1}\left\{\begin{array}{ccc} j_1-1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}+c_0(j_s)\left\{\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}+\frac{c_+(j_s)}{j_1}\left\{\begin{array}{ccc} j_1+1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}$	j^2
6. 	$-C^2(j_1)\left\{\begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array}\right\}$	j^2

TABLE I: The different triple graspings. The labels for the segments are as in (11). In red, the classical configurations; in blue, configurations which are absent in the classical case. The details of the evaluations, as well as the explicit values of the coefficients c_-, c_0 and c_+, are reported in the Appendix. With abuse of notation, we used the same symbol c for the coefficients of types 2, 3 and 5; however, a permutation of the segment labels is involved in going from one type to the other. For each grasping type, a single configuration is shown; the others can be obtained by permutations of the segments. Finally, notice that there are symmetries in the evaluation: for instance in type 2, we chose to write the final result in terms of $\{6j\}$ shifted in the variable j_1, but we could have just as well shifted j_2 or j_6, and the coefficients $c_\pm(j_s)$, $c_0(j_s)$ would have accordingly changed.

IV. THE SEMICLASSICAL LIMIT OF THE VOLUME

All the non–zero graspings (plus all possible permutations) listed in Table I contribute to the quantum volume. However, not all of them contribute to the leading order of the large spin limit. Indeed, the graspings have different overall scalings, which we reported for commodity in the column on the right. In particular we see that all the degenerate (classically absent) graspings do not contribute to the leading order.

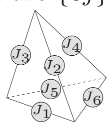
Let us focus our attention to the first non–degenerate grasping, number 2 in the Table. This, we recall, is the only grasping entering the definition (22) of the volume. As we see from the Table, the evaluation of the grasping gives a superposition of $\{6j\}$s. To study the large spin limit, consider first the coefficients: as we show in the Appendix, in the large spin limit we have

$$c_\pm(j_s) \simeq \mp\frac{2}{l_1}A_{123}A_{156}, \tag{29}$$

$$c_0(j_s) \sim j^2. \tag{30}$$

Here $l_1 = j_1 + \frac{1}{2}$ and A_{123} is the area of the triangle bound by the segments l_1, l_2 and l_3. The asymptotics (29) are crucial: if we compare them with (9), we see that we are on the right track, a factor $(\sin\theta_1)/3$ is all that is missing to recover the classical formula of the volume. On the other hand, (30) shows that the "diagonal" term, the one with j_1 unchanged in the $\{6j\}$, can be neglected at the leading order. The leading order then emerges only from the superdiagonal and the subdiagonal, namely the ones with $j_1 \pm 1$ in the $\{6j\}$. Interestingly, this is analogous to what happens for the volume operator in the canonical approach of loop quantum gravity (see for instance [16]).

Using (8) to expand the $\{6j\}$s entering the grasping 2, the result reads



$$\epsilon_{IJK}\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^J_{s_2}}\frac{\delta}{\delta J^K_{s_3}}\Bigg|_{J=0} = \frac{2}{l_1}\frac{A_{123}A_{156}}{\sqrt{12\pi\, V(j_s)}}\Big[\cos\Big(S_{\rm R}(j_1-1)+\frac{\pi}{4}\Big) - \cos\Big(S_{\rm R}(j_1+1)+\frac{\pi}{4}\Big)\Big] + O(j^2). \quad (31)$$

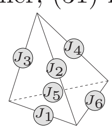
For large spins, the Regge actions can be expanded. Using the well known property $\frac{\partial S_{\rm R}}{\partial j_s} = \theta_s$, we have

$$S_{\rm R}(j_1 \pm 1) \simeq S_{\rm R}(j_s) + \frac{\partial S_{\rm R}}{\partial j_1}\delta j_1 = S_{\rm R}(j_s) \pm \theta_1. \quad (32)$$

Consequently,

$$\cos\Big(S_{\rm R}(j_1-1)+\frac{\pi}{4}\Big) - \cos\Big(S_{\rm R}(j_1+1)+\frac{\pi}{4}\Big) \simeq 2\,\sin\theta_1\,\sin\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big) \quad (33)$$

The expansion of the Regge action has produced the sine of the dihedral angle needed to recover the classical formula for the volume. Putting everything together, (31) reads



$$\frac{i}{3!}\,\epsilon_{IJK}\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^J_{s_2}}\frac{\delta}{\delta J^K_{s_3}}\Bigg|_{J=0} = i\,\frac{V(j_s)}{\sqrt{12\pi\, V(j_s)}}\,\sin\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big) + O(j^2). \quad (34)$$

In the large spin limit, we then have

$$\langle \mathcal{V}_{\tilde\tau}\rangle \sim i\frac{1}{Z}\sum_{j\gg 1}\prod_s d_{j_s}\prod_\tau \{6j\}\, V_{\tilde\tau}(j_s)\,\tan\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big), \quad (35)$$

We see that the leading order of the quantum volume is indeed proportional to the classical formula (9), but there is the extra factor of the tangent of the Regge action. This was to be expected, and it is consistent with the formal manipulation (28).

As discussed above, we can consider the squared volume $\mathcal{V}^2$, given by (minus) the squared of the triple grasping in (24), to obtain a real expectation value and avoid the extra tangent factor. However, graspings in general do not commute, thus the definition of their products requires an ordering prescription. For the objective of studying the semiclassical limit, the appropriate ordering seems to take some sort of "temporal ordering": we act with the two triple graspings one after the other, without allowing them to self–intersect. With this ordering, it is easy to compute the value of $\mathcal{V}^2$ starting from the results listed in Table I. In particular, acting twice in a row with the grasping type 2 and proceeding as above, we obtain the following leading order,

$$\begin{aligned}
&\left(\frac{2}{l_1}A_{123}A_{156}\right)^2\left[\begin{Bmatrix} j_1-2 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix} - 2\begin{Bmatrix} j_1 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix} + \begin{Bmatrix} j_1+2 & j_2 & j_3\\ j_4 & j_5 & j_6\end{Bmatrix}\right] = \\
\simeq\;& \frac{\left(\frac{2}{l_1}A_{123}A_{156}\right)^2}{\sqrt{12\pi\,V(j_s)}}\left[\cos\Big(S_{\rm R}(j_1-2)+\frac{\pi}{4}\Big) - 2\cos\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big) + \cos\Big(S_{\rm R}(j_1+2)+\frac{\pi}{4}\Big)\right] = \\
\simeq\;& -4\frac{\left(\frac{2}{l_1}A_{123}A_{156}\right)^2}{\sqrt{12\pi\,V(j_s)}}\,\sin^2\theta_1\,\cos\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big) \equiv -(3!)^2\frac{V^2(j_s)}{\sqrt{12\pi\,V(j_s)}}\,\cos\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big),
\end{aligned} \quad (36)$$

from which we conclude that

$$\langle \mathcal{V}^2(\tilde\tau)\rangle \sim \frac{1}{Z}\sum_{j\gg 1}\prod_s d_{j_s}\prod_\tau \{6j\}\, V^2_{\tilde\tau}(j_s). \quad (37)$$

We obtain the correct semiclassical limit for the value of the squared volume. Notice that this result can be generalised to arbitrary powers of the triple grasping: this ordering prescription allows to immediately identify the (semiclassical limit of the) n-th power of the triple grasping (which is giving raise to $\pm n$ shifts in the $\{6j\}$ symbol) with the n-th power of the volume (times the factor i tangent if n is odd).

In conclusion, the asymptotics of (powers of) the triple grasping are dominated by (powers of) the classical formula (9) for the volume of a tetrahedron. It is interesting to notice that the polynomial part of this formula arises from the coefficients of the grasping, see (29). The non–polynomial part, namely the sine of the dihedral angle, arises on the other hand from the expansion of the Regge action entering the modified $\{6j\}$s with ± 1. Thus the fact that the triple grasping produces a superposition of $\{6j\}$s is crucial. Furthermore, let us remark again that the fact that the leading asymptotics come from the terms with ± 1 is somewhat reminiscent of the fact that in the spin network basis the canonical 3d volume operator in loop quantum gravity has values only on the superdiagonal and the subdiagonal.

V. QUANTUM CORRECTIONS

Having discussed the meaning of the factor i tangent in (35), we now look at the next to leading order corrections to the quantum volume. Let us first consider the quantum corrections to the grasping of type 2. These are the only ones entering the definition (22) of the volume. There are three types of corrections:

- Corrections from $O(l^2)$ terms in the coefficients $c_i(l_s)$. These can be read from the Appendix, respectively from (B22), (B18) and (B23).

$$\frac{c_-^{(1)}(j_s)}{\sqrt{12\pi\, V(j_s)}}\cos\Big(S_{\rm R}(j_1-1)+\frac{\pi}{4}\Big)+\frac{c_0(j_s)}{\sqrt{12\pi\, V(j_s)}}\cos\Big(S_{\rm R}(j_1)+\frac{\pi}{4}\Big)+\frac{c_+^{(1)}(j_s)}{\sqrt{12\pi\, V(j_s)}}\cos\Big(S_{\rm R}(j_1+1)+\frac{\pi}{4}\Big).$$

  Using the fact that $c^{(1)} := c_-^{(1)} \equiv c_+^{(1)}$ (see the Appendix) and expanding the Regge action as above, this correction can be simply written as

$$\frac{i}{6}\,\frac{2\,c^{(1)}(j_s)\,\cos\theta_1+c_0(j_s)}{\sqrt{12\pi\, V(j_s)}}\cos\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big). \tag{38}$$

  Recall that $c^{(1)}$ and c_0 depend on the configuration chosen (*i.e.* the triplet grasped – here 126). The correction to the volume is obtained summing over all configurations (four different ones using (22), sixteen using (23)), which restores a symmetric expression.

- Corrections from higher orders in the expansion (32) of the Regge action. These can be obtained as follows. First of all, we keep up to the second order term in the expansion of the Regge action,

$$S_{\rm R}(j_1\pm 1)\simeq S_{\rm R}(j_s)\pm\theta_1+G_{11}.$$

  The exact form of the second derivative $G_{11} := \frac{1}{2}\frac{\partial^2 S_{\rm R}}{\partial j_1^2}$ can be computed from elementary geometry (for examples, see [9]). From dimensional analysis, it is clear that $G_{11}\sim 1/j$, thus we expect this term to contribute to the corrections. The contribution can be easily computed. We have $\sin(\theta_1+G_{11})\simeq\sin\theta_1+G_{11}\cos\theta_1$ instead of (33), thus the extra piece gives a correction to (34) of

$$i\,\cot\theta_1\,G_{11}\,\frac{V(j_s)}{\sqrt{12\pi\, V(j_s)}}\sin\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big). \tag{39}$$

  Here again, one has to sum over all the relevant configurations to obtain the correction to the volume.

- Corrections from higher orders in the expansion (8) of the $\{6j\}$ symbol. Unfortunately, the next term in (8) is not known in the literature, thus we cannot pursue this analysis to the end. However, numerical investigations performed in [9] hint for a next term of the type $\frac{N}{V(j_s)^{5/6}}\cos(S_{\rm R}(j_s)+\phi)$, where N and ϕ are numerical constants. This ansatz is just one power of j below the first term, thus it contributes to the next to leading order,

$$\frac{i}{6}\,\frac{\sqrt{12\pi}\,N}{V(j_s)^{5/3}}\,\frac{V(j_s)}{\sqrt{12\pi\, V(j_s)}}\Big[\sin(\phi-\frac{\pi}{4})\cos\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big)+\cos(\phi-\frac{\pi}{4})\sin\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big)\Big]. \tag{40}$$

The relative sign of this correction depends on the sign of the numerical constants N and ϕ. The fit obtained in [9] for the equilateral case when all j_s are the same, gave $N=-0.36$, $\phi=0.68$.

Summing (38), (39) and (40) among themselves and over the four configurations of grasping type 2 (corresponding to grasping the four different vertices), we obtain the overall next to leading order correction to the quantum volume.

The analysis of the corrections of the grasping of type 3 leads exactly to the same results above. However the two definitions (22) and (23) do differ at the level of quantum corrections, as the latter also receives contributions from the degenerate graspings of type 4, 5 and 6.

Consider first the graspings of type 5. Its leading order can be immediately read from type 2 - provided we take into account the extra denominators reducing by one power of j the coefficients of the graspings. We can thus write

$$\frac{i}{6}\,\frac{c_0(j_s)-4A_{123}A_{156}\cos\theta_1/j_1^2}{\sqrt{12\pi\,V(j_s)}}\cos\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big). \tag{41}$$

Finally, 4 and 6 can be recognised, using the results of Section III A, to be the values of the various scalar products between segment vectors, $\ell_s\cdot\ell_{s'}$. Including all the permutations, we simply have the overall correction

$$-\frac{i}{6}\,\frac{\sum_{s,s'}\ell_s\cdot\ell_{s'}}{\sqrt{12\pi\,V(j_s)}}\cos\Big(S_{\rm R}(j_s)+\frac{\pi}{4}\Big). \tag{42}$$

These extra corrections (41) and (42) allow us to distinguish and differentiate between the two definitions of the quantum volume.

VI. CONCLUSIONS

By relating the classical observable to a grasping operator and using the recoupling theory of SU(2), we explicitly computed the value of the quantum volume of a labelled tetrahedron in the PR model. The various terms contributing to the quantum value are reported in Table I, and the details of the evaluation in the Appendix.

Then, we studied the semiclassical limit by considering the large spin expansion. The leading order, given in (35), is indeed dominated by the classical formula, but an additional factor is present. This factor, i times the tangent of the Regge action, can be understood as a consequence of the fact that the PR model sums over both orientations of spacetime, and was indeed anticipated in [4]. Consistently, the value of the squared volume should not have this problem, and we confirmed this expectation: the value of the squared volume, given in (37), is dominated precisely by the classical formula. In doing so, we introduced an ordering prescription for the products of graspings. Using this prescription, it is easy to see that the nth power of the triple grasping has the asymptotics dominated by the nth power of the classical volume (times a factor i tangent if n is odd). The existence of such an ordering prescription could turn out to be important in studying the large spin expansion of spinfoam models of gravity coupled to matter field.

It is interesting to note that a key factor of the classical formula, namely the sine of the dihedral angle, does not come directly from the coefficients of the grasping, but from the fact that the triple grasping changes the initial $\{6j\}$ into a superposition of $\{6j\}$s. This fact mimics the structure of the canonical volume operator.

We considered two different definitions of the quantum volume, related to classically equivalent different discretisations, and the results summarised above hold for both definitions. However, we also showed how one can compute the next to leading order quantum correction, and this is where the two versions of the quantum volume differ. We expect this to be a generic feature of constructing quantum observables starting from discrete classical quantities: different quantum observables corresponding to classically equivalent quantities, even if they have the same semiclassical leading order, could still differ at the next to leading order. The latter can be then used to distinguish them.

In particular, the next to leading order also depends on the second term in the expansion of the $\{6j\}$ symbol, a term which is not analytically known at the moment. At this stage, we are thus not yet able to say what is the relative sign of the correction, and so we leave the issue open. However, let us stress that knowing the next term of the $\{6j\}$ is also useful to study how quantum gravity can give rise to a different perturbative expansion for the graviton propagator, as suggested in [9], so computing this term analytically could be useful for several reasons.

The results reported here show that the large spin limit reproduces semiclassical geometry in 3d quantum gravity. It is important to extend a similar analysis to the physically interesting 4d case. However, the situation is more subtle in

4d. In fact, the classical geometrical quantities do not commute in 4d spinfoam models of quantum gravity. Therefore an eigenstate will not in general have the semiclassical limit, just as well as one can not study the semiclassical limit of the harmonic oscillator on an eigenstate of the energy. It is necessary to first construct appropriate semiclassical states, or coherent states. A class of semiclassical states for the tetrahedron of Loop Quantum Gravity were proposed in [17]. It would be interesting to apply the logic described here to study the value of the quantum volume of these states.

Acknowledgments

We thank Carlo Rovelli for useful discussions on the construction of the volume grasping, and Laurent Freidel for suggestions on the use of the recoupling theory.

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APPENDIX A: EVALUATION OF SU(2) SPIN NETWORKS

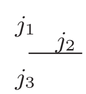
A spin network s is a triple $\{\gamma, j_l, i_n\}$ consisting of a graph γ; a set of irreps j_l aassociated to the links; a set of intertwiners i_n associated to the nodes. A spin network state $\psi_s(g_l)$ is constructed assigning a group element g_l to each link in the corresponding irrep j_l, namely assigning a representation matrix $D^{(j_l)}(g_l)$, and then contracting the indices of all the matrices with the intertwiners on the nodes:

$$\psi_s(g_l) = \prod_l D^{(j_l)}(g_l) \prod_n i_n. \tag{A1}$$

By construction, this is a gauge–invariant quantity, belonging to the space $\mathrm{Inv}\Big[\bigotimes_l \mathcal{H}_{j_l}\Big]$. Upon the interpretation of the group elements as parallel transports (3), (A1) becomes a cylindrical function of the connection [18, 19]. The spin networks span the kinematical Hilbert space of LQG, which is given by

$$L^2[\mathcal{A}/\mathcal{G}] \cong \bigoplus_j \bigotimes_{\mathrm{v}} \mathrm{Inv}\left[\bigotimes_l \mathcal{H}_{j_l}\right]. \tag{A2}$$

Evaluating a spin network refers to taking all group elements to the identity, $g_l \mapsto \mathbb{1}\ \forall l$, and defines a map $L^2[\mathcal{A}/\mathcal{G}] \mapsto \mathbb{C}$.

To fix the normalisation of the spin networks, consider the θ graph 

. Using the normalised Wigner $3m$-coefficients as intertwiners, (A1) reads

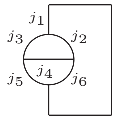
$$\psi(g_1, g_2, g_3) = D^{(j_1)}_{m_1 n_1}(g_1) D^{(j_2)}_{m_2 n_2}(g_2) D^{(j_3)}_{m_3 n_3}(g_3) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}, \tag{A3}$$

and its evaluation gives

$$\psi(\mathbb{1}, \mathbb{1}, \mathbb{1}) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \begin{cases} 1 & \text{if } |j_1 - j_2| \leq j_3 \leq j_1 + j_2, \\ 0 & \text{otherwise.} \end{cases} \tag{A4}$$

The inequality that has to be satisfied is the usual Clebsch–Gordan condition. Given an arbitrary spin network, its evaluation is identically zero unless the Clebsch–Gordan condition hold at all nodes.

1. The $\{6j\}$ symbol.

On the tetrahedral graph 

, the evaluation of the spin network gives the $\{6j\}$ symbol defined in (7),

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} := \left(\begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left(\begin{array}{ccc} j_1 & j_5 & j_6 \\ m_1 & m_5 & m_6 \end{array} \right) \left(\begin{array}{ccc} j_4 & j_5 & j_3 \\ m_4 & m_5 & m_3 \end{array} \right) \left(\begin{array}{ccc} j_4 & j_2 & j_6 \\ m_4 & m_2 & m_6 \end{array} \right).$$

This object has the symmetries associated with the geometrical symmetries of the tetrahedron pictured above and, more fundamentally, it satisfies the Biedenharn–Elliott identity,

$$\left\{ \begin{array}{ccc} j_1 & k_2 & j_3 \\ k_1 & j_2 & k_3 \end{array} \right\} \left\{ \begin{array}{ccc} l_1 & k_2 & l_3 \\ k_1 & l_2 & k_3 \end{array} \right\} = \sum_j d_j \left\{ \begin{array}{ccc} j_1 & j_2 & k_3 \\ l_2 & l_1 & j \end{array} \right\} \left\{ \begin{array}{ccc} j_2 & j_3 & k_1 \\ l_3 & l_2 & j \end{array} \right\} \left\{ \begin{array}{ccc} j_3 & j_1 & k_2 \\ l_1 & l_3 & j \end{array} \right\}. \tag{A5}$$

An analytic expression for the $\{6j\}$ is provided by the Racah formula [21],

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} = (-1)^{j_1+j_2+j_3+j_4} \sqrt{\Delta(j_1,j_2,j_3)\Delta(j_1,j_5,j_6)\Delta(j_4,j_5,j_3)\Delta(j_4,j_2,j_6)} \sum_k \frac{(-1)^k (k+1)!}{f(k)}, \tag{A6}$$

where

$$\Delta(j_1,j_2,j_3) := \frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!} \tag{A7}$$

and

$$\begin{aligned} f(k) \ := \ & (k-j_1-j_2-j_3)!(k-j_1-j_5-j_6)!(k-j_4-j_2-j_6)! \times \\ & \times (k-j_4-j_5-j_3)!(j_1+j_2+j_4+j_5-k)! \times \\ & \times (j_2+j_3+j_5+j_6-k)!(j_1+j_3+j_4+j_6-k)! \end{aligned} \tag{A8}$$

k is an integer, and the sum in (A6) is over all admissable k. From the graphical representation of the $\{6j\}$ symbol, it is clear that the Clebsch–Gordan conditions must hold on all four nodes, so that the functions Δ appearing in (A6) are all well–defined.

Some useful explicit values of (A6) are the followings:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ 1 & j_3 & j_2 \end{array} \right\} = \frac{1}{2} \frac{C^2(j_2)+C^2(j_3)-C^2(j_1)}{\sqrt{d_{j_2}\ C^2(j_2)\ d_{j_3}\ C^2(j_3)}}, \tag{A9}$$

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ 1 & j_3 & j_2+1 \end{array} \right\} = \frac{1}{2} \frac{\sqrt{(1+j_2+j_3-j_1)(1+j_2-j_3+j_1)(-j_2+j_3+j_1)(2+j_2+j_3+j_1)}}{\sqrt{(j_2+1)\, d_{j_3}\, d_{j_2}\, d_{j_2+1}\, C^2(j_3)}}, \tag{A10}$$

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ 1 & j_3 & j_2-1 \end{array} \right\} = -\frac{1}{2} \frac{\sqrt{(j_2+j_3-j_1)(j_2-j_3+j_1)(1-j_2+j_3+j_1)(1+j_2+j_3+j_1)}}{\sqrt{j_2\, d_{j_3}\, d_{j_2}\, d_{j_2-1}\, C^2(j_3)}}, \tag{A11}$$

from which we immediately read

$$\left\{ \begin{array}{ccc} j & j & 1 \\ 1 & 1 & j \end{array} \right\} = \frac{1}{\sqrt{6\, d_j\, C^2(j)}}, \qquad \left\{ \begin{array}{ccc} j & j & 1 \\ 1 & 1 & j+1 \end{array} \right\} = \frac{j}{\sqrt{6\, d_j\, C^2(j)}}, \qquad \left\{ \begin{array}{ccc} j & j & 1 \\ 1 & 1 & j-1 \end{array} \right\} = -\frac{j+1}{\sqrt{6\, d_j\, C^2(j)}}. \tag{A12}$$

As it will be useful in the following, let us recall here the definition of the {9j} symbol,

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{array} \right\} = \sum_j d_j \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_6 & j_9 & j \end{array} \right\} \left\{ \begin{array}{ccc} j_4 & j_5 & j_6 \\ j_2 & j & j_8 \end{array} \right\} \left\{ \begin{array}{ccc} j_7 & j_8 & j_9 \\ j & j_1 & j_4 \end{array} \right\}. \tag{A13}$$

Notice that it is antisymmetric under exchange of rows, thus

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \\ j_7 & j_8 & j_9 \end{Bmatrix} \equiv 0. \tag{A14}$$

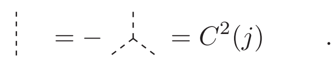
The {9j} symbol is the evaluation of the spin network on the hexagonal graph.

APPENDIX B: GRASPING RULES AND RECOUPLING THEORY

The action of an algebra derivative on a group element in a representation j is given by

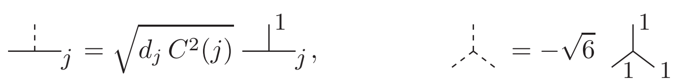
$$\frac{\delta}{\delta J^I} D^{(j)}(e^J)\Big|_{J=0} = -iT^{I(j)}, \tag{B1}$$

where $T^{I(j)}$ is the I-th generator, in the irrep j. We picture this action as $(-i)$ j. The dashed line represents the insertion of the algebra generator. A dashed 3-valent vertex represents the algebra structure constants $i\epsilon_{ijk}$. The insertion of an algebra generator is equivalent to adding a link in the adjoint irrep, up to normalisation. We fix the normalisation[1] as to match the grasping rules of Bar–Natan [22], namely



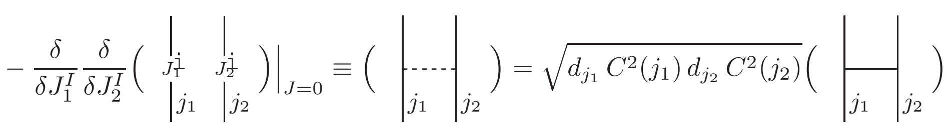
(B2)

To this end, we choose



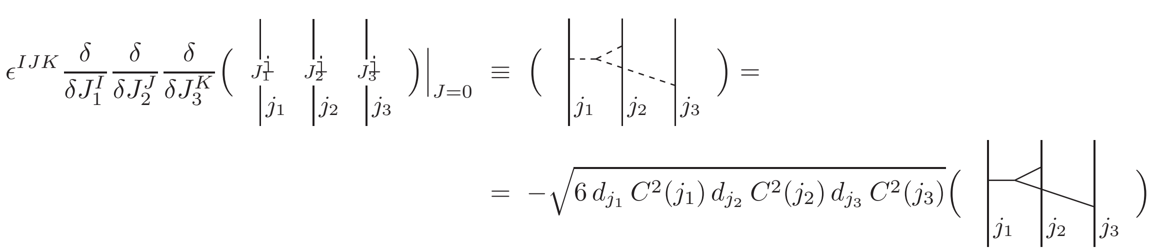
(B3)

With this normalization, the action of the double grasping is the insertion in the spin network of an additional link in the adjoint irrep, weighted by the labels of the two links grasped:



(B4)

Here and in the following, a line with no labels means a link coloured with the adjoint $j = 1$ irrep. In the same way, the triple grasping is the insertion of a 3-valent node in the adjoint irrep, weighted by the labels of the three links grasped:



(B5)

Therefore, evaluating the action of a grasping operator on a given spin network amounts to evaluating a new spin network, obtained by adding to the former the graph of the grasp in the adjoint irrep. To compute the action of the grasping operators, it is then convenient the use of the recoupling theory.

The key equation to be used is the recoupling theorem:

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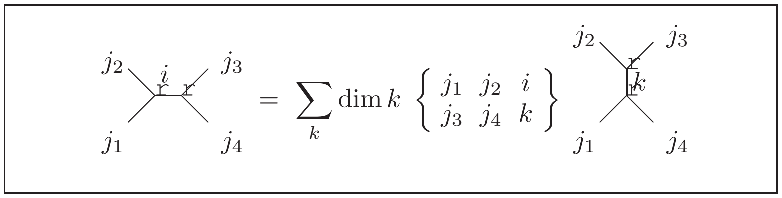
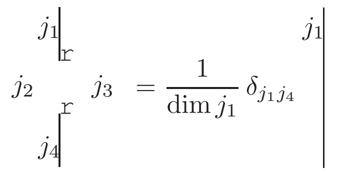
[1] For a different normalisation, see for instance [23]



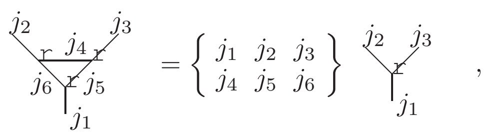
$$= \sum_k \dim k \left\{ \begin{array}{ccc} j_1 & j_2 & i \\ j_3 & j_4 & k \end{array} \right\} \qquad \text{(B6)}$$

From the definition of the θ spin network and the Clebsch–Gordan conditions, it follows that



$$= \frac{1}{\dim j_1} \delta_{j_1 j_4} \qquad \text{(B7)}$$

Using (B6) repeatedly, it is easy to demonstrate the following 3-vertex contraction:



$$= \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \qquad \text{(B8)}$$

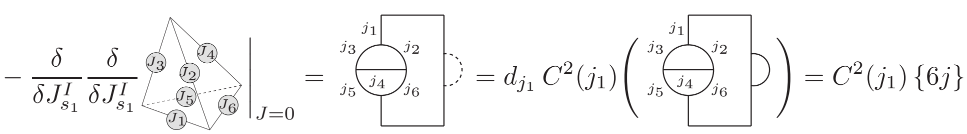
Notice that if we add an extra node to close the links j_1, j_2 and j_3 on both sides of (B8), and using the normalisation of the θ spin network, we get back the definition of the $\{6j\}$ symbol as the evaluation of the tetrahedral spin network.

To evaluate the graspings, the strategy is the following: we use the definitions (B4) and (B5) to obtain new graphs; then we use (B6) enough times to reduce the new graphs to the original ones. In particular in the next Sections, we will apply this strategy to the tetrahedral graph, in order to study the spectra of geometrical observables in the Ponzano–Regge model.

1. The double grasping on the $\{6j\}$

Here we show how to evaluate the double grasping on the tetrahedral spin network. These results were used in Section III A.

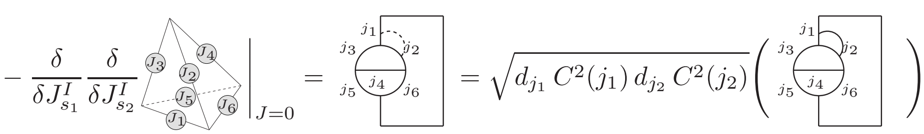
Computing the quadratic grasping on a single link is trivial, all we need is the bubble move (B7):



$$- \frac{\delta}{\delta J^I_{s_1}} \frac{\delta}{\delta J^I_{s_1}} \Big|_{J=0} = d_{j_1}\, C^2(j_1) \left(\;\right) = C^2(j_1)\,\{6j\}. \qquad \text{(B9)}$$

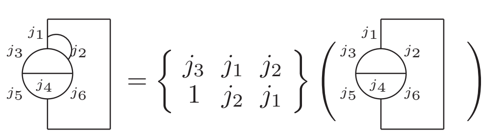
This proves (16) used to compute the spectrum of lengths.

Computing the grasping on two different segments sharing a point is also very simple. We have



$$- \frac{\delta}{\delta J^I_{s_1}} \frac{\delta}{\delta J^I_{s_2}} \Big|_{J=0} = \sqrt{d_{j_1}\, C^2(j_1)\, d_{j_2}\, C^2(j_2)} \left(\;\right). \qquad \text{(B10)}$$

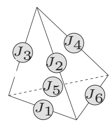
The graph obtained with the grasping can be straightforwardly reduced to the initial tetrahedral graph, simply using (B8) once:



$$= \left\{ \begin{array}{ccc} j_3 & j_1 & j_2 \\ 1 & j_2 & j_1 \end{array} \right\} \left(\;\right), \qquad \text{(B11)}$$

Therefore, using the explicit expression (A9), we obtain (18),

$$-\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^I_{s_2}}$$



$$\Bigg|_{J=0} = \frac{1}{2}[C^2(j_1)+C^2(j_2)-C^2(j_3)]\,\{6j\}, \tag{B12}$$

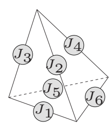
which we used to compute the spectrum of angles.

2. The triple grasping on the $\{6j\}$

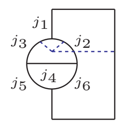
Here we show how to evaluate the triple grasping on the $\{6j\}$ symbol, a result used in Section III B. As shown in Table I, there are several types of graspings that one has to consider. We proceed in the order given there.

Grasping 1.

$$\left(\epsilon_{IJK}\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^J_{s_2}}\frac{\delta}{\delta J^K_{s_3}}\right)$$



$$\Bigg|_{J=0} =$$



$$=$$

$$= -\sqrt{6\, d_{j_1}\, C^2(j_1)\, d_{j_2}\, C^2(j_2)\, d_{j_3}\, C^2(j_3)}\sum_k d_k \begin{Bmatrix} j_2 & j_1 & j_3 \\ 1 & j_3 & k \end{Bmatrix}\begin{Bmatrix} j_3 & j_1 & j_2 \\ 1 & j_2 & k \end{Bmatrix}\begin{Bmatrix} j_1 & j_1 & 1 \\ 1 & 1 & k \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \tag{B13}$$

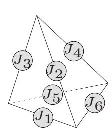
If we use the symmetries of the $\{6j\}$ symbol, we see that the coefficient obtained in front of the original $\{6j\}$ matches the definition of the $\{9j\}$ symbol introduced in (A13),

$$\sum_k d_k \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_3 & 1 & k \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_2 & k & 1 \end{Bmatrix}\begin{Bmatrix} 1 & 1 & 1 \\ k & j_1 & j_1 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \\ 1 & 1 & 1 \end{Bmatrix} \equiv 0. \tag{B14}$$

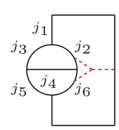
By symmetry, this result applies to all coplanar triples, and we recover the classical result: there is no contribution to the volume from coplanar segments.

Grasping 2.

$$\left(\epsilon_{IJK}\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^J_{s_2}}\frac{\delta}{\delta J^K_{s_6}}\right)$$



$$\Bigg|_{J=0} =$$



$$=$$

$$= -\sqrt{6\, d_{j_1}\, C^2(j_1)\, d_{j_2}\, C^2(j_2)\, d_{j_6}\, C^2(j_6)}\sum_k d_k \begin{Bmatrix} j_3 & j_1 & j_2 \\ 1 & j_2 & k \end{Bmatrix}\begin{Bmatrix} j_5 & j_1 & j_6 \\ 1 & j_6 & k \end{Bmatrix}\begin{Bmatrix} j_1 & j_1 & 1 \\ 1 & 1 & k \end{Bmatrix}\begin{Bmatrix} k & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \tag{B15}$$

The calculation yields substantial differences from the previous one. The main one is the new spin k replacing the original j_1 in the $\{6j\}$ (by symmetry, we could have replaced j_2 or j_6, without changing the final result). From the third $\{6j\}$ in the coefficient above, we read that the sum ranges over $k = j_1 - 1, j_1, j_1 + 1$. We can thus write the result above as

$$c_-(j_s)\begin{Bmatrix} j_1 - 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} + c_0(j_s)\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} + c_+(j_s)\begin{Bmatrix} j_1 + 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}, \tag{B16}$$

where, using the formulae reported in the Appendix A,

$$\begin{aligned} c_-(j_s) &= \frac{(j_1+1)}{4j_1(2j_1+1)}\sqrt{(j_1+j_2-j_3)(j_1-j_2+j_3)(1-j_1+j_2+j_3)(1+j_1+j_2+j_3)}\times \\ &\qquad \sqrt{(j_1+j_5-j_6)(j_1-j_5+j_6)(1-j_1+j_5+j_6)(1+j_1+j_5+j_6)}, \end{aligned} \tag{B17}$$

$$c_0(j_s) = -\frac{[C^2(j_1)+C^2(j_2)-C^2(j_3)][C^2(j_1)+C^2(j_6)-C^2(j_5)]}{4\,C^2(j_1)}, \tag{B18}$$

$$\begin{aligned} c_+(j_s) &= -\frac{j_1}{4(j_1+1)(2j_1+1)}\sqrt{(1+j_1+j_2-j_3)(1+j_1-j_2+j_3)(-j_1+j_2+j_3)(2+j_1+j_2+j_3)}\times \\ &\qquad \sqrt{(1+j_1+j_5-j_6)(1+j_1-j_5+j_6)(-j_1+j_5+j_6)(2+j_1+j_5+j_6)}. \end{aligned} \tag{B19}$$

Let us now discuss the large spin expansion of this result. First of all, recall that the spins are related to the (here adimensional) lengths through $l_s = j_s + \frac{1}{2}$. Second, let us recall Heron's formula [20] for the area of a triangle with side lengths a, b, c:

$$A_{abc} = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}. \tag{B20}$$

As a useful shorthand, we will define the quantity

$$V_1 = \frac{2}{l_1}A_{l_1l_2l_3}A_{l_1l_5l_6}, \tag{B21}$$

and generalize to arbitrary V_i by the symmetry of the tetrahedron. Notice that from (9) we immediately have $V = \frac{1}{3}V_i\sin\theta_i$.

The coefficients of the functions in the lengths $c_\pm(l_s)$ are polynomials of order three. To second order, we have

$$\begin{aligned} c_-(l_s) &\simeq V_1\left[\left(1+\frac{1}{l_1}\right)\left(1-\frac{1}{4(l_1+l_2-l_3)}-\frac{1}{4(l_1-l_2+l_3)}+\frac{1}{4(-l_1+l_2+l_3)}-\frac{1}{4(l_1+l_2+l_3)}\right)\times\right. \\ &\qquad \left.\left(1-\frac{1}{4(l_1+l_5-l_6)}-\frac{1}{4(l_1-l_5+l_6)}+\frac{1}{4(-l_1+l_5+l_6)}-\frac{1}{4(l_1+l_5+l_6)}\right)+o\left(\frac{1}{l^2}\right)\right], \end{aligned} \tag{B22}$$

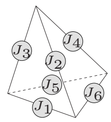
$$\begin{aligned} c_+(l_s) &\simeq -V_1\left[\left(1-\frac{1}{l_1}\right)\left(1+\frac{1}{4(l_1+l_2-l_3)}+\frac{1}{4(l_1-l_2+l_3)}-\frac{1}{4(-l_1+l_2+l_3)}+\frac{1}{4(l_1+l_2+l_3)}\right)\times\right. \\ &\qquad \left.\left(1+\frac{1}{4(l_1+l_5-l_6)}+\frac{1}{4(l_1-l_5+l_6)}-\frac{1}{4(-l_1+l_5+l_6)}+\frac{1}{4(l_1+l_5+l_6)}\right)+o\left(\frac{1}{l^2}\right)\right] \end{aligned} \tag{B23}$$

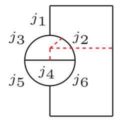
Let us define the coefficients of the expansion as in $c_\pm(j_s) = c_\pm^{(0)}(j_s) + c_\pm^{(1)}(j_s) + \ldots$ From the equations above, we see that $c_\pm^{(0)}(j_s) = \mp V_1$, and $c_-^{(1)}(j_s) \equiv c_+^{(1)}(j_s)$.

All terms in $c_0(j_s)$, on the other hand, are of order j^2. In particular, notice that $c_0(j_s) = -(\ell_1\cdot\ell_2)\,(\ell_1\cdot\ell_6)/\ell_1^2$.

The values from the other three vertex graspings can be obtained by symmetry. The other three contributions can be obtained as above, considering the other three cases (j_1, j_3, j_5), (j_2, j_3, j_4) and (j_4, j_5, j_6).

Grasping 3.

$$\left(\epsilon_{IJK}\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^J_{s_2}}\frac{\delta}{\delta J^K_{s_4}}\right)$$


$\Big|_{J=0} =$


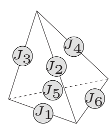
$=$

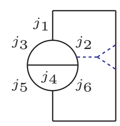
$$= -\sqrt{6\, d_{j_1}\, C^2(j_1)\, d_{j_2}\, C^2(j_2)\, d_{j_4}\, C^2(j_4)}\sum_k d_k \begin{Bmatrix} j_3 & j_2 & j_1 \\ 1 & j_1 & k \end{Bmatrix}\begin{Bmatrix} j_6 & j_2 & j_4 \\ 1 & j_4 & k \end{Bmatrix}\begin{Bmatrix} j_2 & j_2 & 1 \\ 1 & 1 & k \end{Bmatrix}\begin{Bmatrix} j_1 & k & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \tag{B24}$$

Under the permutation $123456 \mapsto 213645$, (B24) is equivalent to (B15). Therefore, this grasping reproduces the same results as the previous one. Notice that this is in agreement with the classical result,

$$V = \frac{1}{3!}\, e_1 \wedge e_2 \wedge e_6 = \frac{1}{3!}\, e_1 \wedge e_2 \wedge e_4. \tag{B25}$$

Grasping 4.

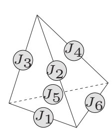
$$\left(\epsilon_{IJK}\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^J_{s_1}}\frac{\delta}{\delta J^K_{s_2}}\right)$$


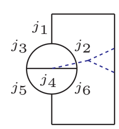
$\Big|_{J=0} =$


$=$

$$\begin{aligned} &= -\sqrt{6\, d_{j_2}\, C^2(j_2)}\, d_{j_1}\, C^2(j_1)\begin{Bmatrix} j_3 & j_1 & j_2 \\ 1 & j_2 & j_1 \end{Bmatrix}\begin{Bmatrix} j_1 & j_1 & 1 \\ 1 & 1 & j_1 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \\ &= -\frac{1}{2}\left[C^2(j_1) + C^2(j_2) - C^2(j_3)\right]\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \end{aligned} \tag{B26}$$

Grasping 5.

$$\left(\epsilon_{IJK}\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^J_{s_1}}\frac{\delta}{\delta J^K_{s_4}}\right)$$


$\Big|_{J=0} =$


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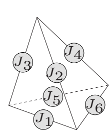
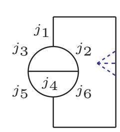
$$\begin{aligned} &-\sqrt{6\, d_{j_1}\, C^2(j_1)\, d_{j_1}\, C^2(j_1)\, d_{j_4}\, C^2(j_4)}\sum_k d_k \begin{Bmatrix} j_2 & j_6 & j_4 \\ 1 & j_4 & k \end{Bmatrix}\begin{Bmatrix} j_5 & j_6 & j_1 \\ 1 & j_1 & k \end{Bmatrix}\begin{Bmatrix} j_1 & j_1 & 1 \\ 1 & 1 & j_1 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & k \end{Bmatrix} = \\ &= \frac{c_-(j_s)}{j_1+1}\begin{Bmatrix} j_1 - 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} + c_0(j_s)\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} + \frac{c_+(j_s)}{j_1}\begin{Bmatrix} j_1 + 1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \end{aligned} \tag{B27}$$

Grasping 6.

$$\left(\epsilon_{IJK}\frac{\delta}{\delta J^I_{s_1}}\frac{\delta}{\delta J^J_{s_1}}\frac{\delta}{\delta J^K_{s_1}}\right)$$


$\Big|_{J=0} =$


$=$

$$= -\sqrt{6}\,\left[d_{j_1}\, C^2(j_1)\right]^{\frac{3}{2}}\frac{1}{d_{j_1}}\begin{Bmatrix} j_1 & j_1 & 1 \\ 1 & 1 & j_1 \end{Bmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = -C^2(j_1)\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}. \tag{B28}$$

[1] C. Rovelli. *Quantum Gravity*. (Cambridge University Press, Cambridge 2004.)
[2] G. Ponzano, T. Regge. “Semiclassical limit of Racah coefficients”, in *Spectroscopy and group theoretical methods in Physics*, F. Bloch ed. (North-Holland, Amsterdam, 1968).
[3] L. Freidel and K. Krasnov, “Spin foam models and the classical action principle,” Adv. Theor. Math. Phys. **2** (1999) 1183 [arXiv:hep-th/9807092].
[4] L. Freidel and K. Krasnov, “Discrete space-time volume for 3-dimensional BF theory and quantum gravity,” Class. Quant. Grav. **16** (1999) 351 [arXiv:hep-th/9804185].
[5] W. Fairbairn, “Fermions in three-dimensional spinfoam quantum gravity,” arXiv:gr-qc/0609040.
[6] L. Freidel, C. Rovelli and S. Speziale, “Perturbative expansion for 3d Yang–Mills theory coupled to quantum gravity in the spinfoam formalism,” in preparation.
[7] C. Rovelli, “Graviton propagator from background-independent quantum gravity,” Phys. Rev. Lett. **97** (2006) 151301 [arXiv:gr-qc/0508124].
E. Bianchi, L. Modesto, C. Rovelli and S. Speziale, “Graviton propagator in loop quantum gravity,” Class. Quant. Grav. **23** (2006) 6989 [arXiv:gr-qc/0604044].
E. R. Livine and S. Speziale, “Group integral techniques for the spinfoam graviton propagator,” to appear on JHEP [arXiv:gr-qc/0608131].
[8] S. Speziale, “Towards the graviton from spinfoams: The 3d toy model,” JHEP **05** (2006) 039 [arXiv:gr-qc/0512102].
[9] E. Livine, S. Speziale and J. Willis “Towards the graviton from spinfoams: higher order corrections in the 3d toy model,” arXiv:gr-qc/0605123.
[10] E. R. Livine and D. R. Terno, “Reconstructing quantum geometry from quantum information: Area renormalisation, coarse-graining and entanglement on spin networks,” arXiv:gr-qc/0603008.
[11] L. Freidel and E. R. Livine, “3d Quantum Gravity And Effective Noncommutative Quantum Field Theory,” Phys. Rev. Lett. **96** (2006) 221301.
[12] A. Baratin and L. Freidel, “Hidden quantum gravity in 3d Feynman diagrams,” arXiv:gr-qc/0604016.
[13] L. Freidel and D. Louapre, “Ponzano-Regge model revisited. I: Gauge fixing, observables and interacting spinning particles,” Class. Quant. Grav. **21** (2004) 5685 [arXiv:hep-th/0401076].
[14] J.D.Roberts. “Classical 6j-symbols and the tetrahedron.” Geometry and Topology 3 (1999), 21–66. [math-ph/9812013]
[15] L. Freidel and D. Louapre, “Asymptotics of 6j and 10j symbols,” Class. Quant. Grav. **20** (2003) 1267 [arXiv:hep-th/0209134].
J. W. Barrett and C. M. Steele, “Asymptotics of Relativistic Spin Networks”, Class.Quant.Grav. **20** (2003) 1341-1362, [arXiv:gr-qc/0209023].
[16] J. Brunnemann and T. Thiemann, “Simplification of the spectral analysis of the volume operator in loop quantum gravity,” Class. Quant. Grav. **23** (2006) 1289 [arXiv:gr-qc/0405060].
[17] C. Rovelli and S. Speziale, “A semiclassical tetrahedron,” Class. Quant. Grav. **23** (2006) 5861 - 5870, [arXiv:gr-qc/0606074].
[18] A. Ashtekar and J. Lewandowski, “Projective Techniques And Functional Integration For Gauge Theories,” J. Math. Phys. **36** (1995) 2170 [arXiv:gr-qc/9411046].
[19] J. C. Baez, “Spin network states in gauge theory,” Adv. Math. **117**, 253 (1996) [arXiv:gr-qc/9411007].
[20] Heron of Alexandria, *Metrica*, (I cen. AD).
[21] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonsky, *Quantum Theory of Angular Momentum* (World Scientific, Singapore 1988).
[22] D. Bar-Natan, “On the Vassiliev Knot Invariants,” Topology **34** (1995) 423
[23] R. De Pietri, “On the relation between the connection and the loop representation of quantum gravity,” Class. Quant. Grav. **14** (1997) 53 [arXiv:gr-qc/9605064].